Today:

- HW 8 comments
- Cholesky factorization
- Applications

HW 8

2) If A is SPD, then all eigenvalues of A are positive.

Let v be a unit eigenvector of A with e-val λ, consider

$$\langle Av, v \rangle$$

Cholesky factorization

- write SPD matrix A as

$$A = LL^T$$

Ex/

$$A = \begin{pmatrix} 1 & 3 & -2 \\ 3 & 25 & -2 \\ -2 & -2 & 30 \end{pmatrix}$$

Find the Cholesky factorization of A.

Soln

$$\begin{pmatrix} 1 & 3 & -2 \\ 3 & 25 & -2 \\ -2 & -2 & 30 \end{pmatrix} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}}_{L^T}$$

$A_{11} = 1 = (l_{11})^2 \Rightarrow l_{11} = 1$ (choice)

$A_{21} = 3 = l_{21} l_{11} \Rightarrow l_{21} = 3$

$A_{31} = -2 = l_{31} l_{11} \Rightarrow l_{31} = -2$

$A_{22} = 25 = (l_{21})^2 + (l_{22})^2 \Rightarrow l_{22} = 4$

$A_{32} = -2 = l_{31} l_{21} + l_{32} l_{22} \Rightarrow l_{32} = 1$

$A_{33} = 30 = (l_{31})^2 + (l_{32})^2 + (l_{33})^2$

$\Rightarrow l_{33} = 5$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 4 & 0 \\ -2 & 1 & 5 \end{pmatrix}$$

Note: choice of signs for l_{11}, l_{22}, l_{33}

A SPD $\iff$ $\lambda > 0$

What if A isn't SPD?

Ex/

$$A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 4 \\ -3 & 4 & 2 \end{pmatrix}$$

What happens if we try to compute the Cholesky factorization of A?

$\det(A) = -4 - 16 - 3(-6) = -2$

$\Rightarrow$ atleast one e-val is negative

$A_{11} = 1 \Rightarrow l_{11} = 1$

$A_{21} = 0 \Rightarrow l_{21} = 0$

$A_{22} = -2 \Rightarrow (l_{21})^2 + (l_{22})^2 = -2$

$\Downarrow$

Applications:

$A = LL^T$, to solve $Ax = b$,

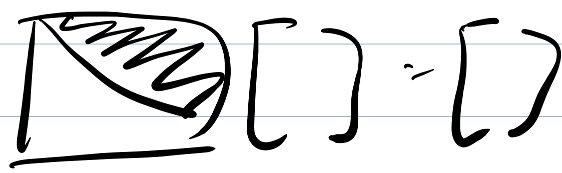
$$L\underbrace{L^T x}_{y} = b$$

① solve $Ly = b$ for y

(forward substitution)

② Solve $L^T x = y$ for x

(back substitution)



More concrete application:

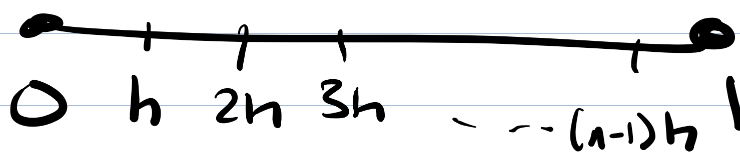
- Numerically solve ODEs
- Ex:

$$-u''(x) = \underbrace{f(x)}_{\hookrightarrow \text{given}}$$

- Centered difference approximation

$$u''(x) \approx \frac{u(x-h) - 2u(x) + u(x+h)}{h^2}$$

- Discretize domain:



Linear system:

$$-u''(x) = f(x)$$

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & & \\ -1 & 2 & -1 & 0 & \cdots & \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}}_{\text{centered difference approximation}} \underbrace{\begin{bmatrix} u(0) \\ u(h) \\ u(2h) \\ \vdots \\ u(1) \end{bmatrix}}_{\text{unknown}} = \underbrace{\begin{bmatrix} f(0) \\ f(h) \\ \vdots \\ f(1) \end{bmatrix}}_{\text{known, given}}$$

$$Au = f$$