Today:

- HW comments
- Floating point considerations
- Lagrange polynomials

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[4]

Let $p \in [a,b]$ be the root of $f \in C^1([a,b])$, and assume $f'(p) \neq f'(p_0)$ for some $p_0 \in [a,b]$. Consider an iteration scheme that is similar to, but different from Newton's method: given p_0, define

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_0)}, \qquad n \geq 0.$$

Assuming that the iterative scheme converges, i.e. that $p_n \to p$ as $n \to \infty$, show that this method has order of convergence $\alpha = 1$.

(*Note:* to actually show that indeed the scheme converges, one needs to assume $f \in C^2([a,b])$. Also, this is a sufficient, but not necessary condition.)

$$g(x) = x - \frac{f(x)}{f'(p_0)}$$

Taylor expansion of $g(p_n)$ around p

$$g(p_n) = g(p) + g'(\xi_n)(p_n - p)$$

$$p_{n+1} = p + g'(\xi_n)(p_n - p)$$

$\xi_n \to p$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lim_{n\to\infty} |g'(\xi_n)|$$

$= |g'(p)| > 0$

$$g'(x) = 1 - \frac{f'(x)}{f'(p_0)}$$

(since $f'(p) \neq f'(p_0)$)

$$g'(p) = 1 - \frac{f'(p)}{f'(p_0)} \neq 0$$

HW 4

1) $1 - \cos(x)$ $\quad |x|$ very close to 0

$\Rightarrow$ "multiply by 1", get rid of $-$

2) a) Taylor expansion of $\sin(\pi/2 + \delta)$ centered at $\pi/2$

$\Rightarrow$ don't need too many terms

b) orders of magnitude,

$10^{-1}, 10^{-2}, \ldots$

4) Apply rounding after every operation

3, 4): error is absolute error

Floating Point Considerations

Ex/ Assume that machine precision is $\varepsilon = 10^{-\alpha}$ for some positive integer α. Consider evaluating the quantity $1 - \cos(\pi/2 + \delta)$ where $0 < \delta << 1$. Use a Taylor expansion to find the largest (approximate) value of δ such that

$$fl(1 - \cos(\pi/2)) = fl(1 - \cos(\pi/2 + \delta))$$

<u>Soln</u> ε is largest # such that

$$fl(1 + \varepsilon) = 1$$

$$1 - \cos(\pi/2) = 1$$

Taylor:

$$1 - \cos(\pi/2 + \delta) = 1 - \cos(\pi/2) + \delta(\sin(\pi/2)) + \delta^2 \text{ terms}, \ldots$$

By Taylor,

$$1 - \cos(\pi/2 + \delta) = 1 - \cos(\pi/2) + \delta(\sin(\xi))$$

$$1 - \cos(\pi/2) + \delta(\sin(\xi)) \leq 1 + \varepsilon$$

$$\delta(\sin(\xi)) \leq \varepsilon$$

$\delta = \varepsilon$ (upper bound)

$\delta := 10^{-d}$

Lagrange polynomials

Given: $x_0, \ldots, x_n$
$f(x_0), \ldots, f(x_n)$

Goal: polynomial of degree n, $P_n(x)$,

$$P_n(x_i) = f(x_i) \quad , i = 0, \ldots, n$$

Lagrange polynomial:

$$P_n(x) = \sum_{i=0}^{n} \underbrace{f(x_i)}_{\text{number}} \underbrace{L_i(x)}_{\text{polynomial}}$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$L_i(x_i) = 1 \qquad L_i(x_j) = 0 \quad , j \neq i$$

Ex1 Given: $x_0 = 1$
$x_1 = 2$
$x_2 = 3$
$f(x) = \sin(x)$

a) Construct a degree 1 interpolating polynomial for this data and use it to estimate $f(1.5)$. Calculate the absolute error.

b) Repeat with a degree 2 interpolating polynomial.

Soln

a) Best choice: x_0, x_1

$$P_1(x) = \sin(1)\left(\frac{x-2}{1-2}\right) + \sin(2)\left(\frac{x-1}{2-1}\right)$$

$$= -\sin(1)(x-2) + \sin(2)(x-1)$$

$|P_1(1.5) - \sin(1.5)| = 0.1221$

Use x_1, x_2, error is 0.2959

Use x_0, x_2, error is 0.3311

b)

$$P_2(x) = \sin(1)\left(\frac{(x-2)(x-3)}{(1-2)(1-3)}\right)$$

$$+ \sin(2)\left(\frac{(x-1)(x-3)}{(2-1)(2-3)}\right)$$

$$+ \sin(3)\left(\frac{(x-1)(x-2)}{(3-1)(3-2)}\right)$$

$$= \frac{1}{2}\sin(1)(x-2)(x-3)$$

$$- \sin(2)(x-1)(x-3)$$

$$+ \frac{1}{2}\sin(3)(x-1)(x-2)$$

$$|P_2(1.5) - \sin(1.5)| = 0.0176$$ 🙂