Today:

- Using interpolation error bounds (HW4)
- Cubic splines

---

[4]

Suppose that $f \in C^2([x_0, x_1])$ for $x_0 < x_1$, and let $P(x)$ be the linear interpolant for f at x_0 and x_1. Using the theorem given in class on the error in polynomial interpolation, derive the following bound:

$$|f(x) - P(x)| \le \frac{1}{8} h^2 \max_{x \in [x_0, x_1]} |f''(x)|,$$

where $h = x_1 - x_0$.

$$f(x) = P(x) + \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1)$$

$$|f(x) - P(x)| = \left| \frac{f''(\xi(x))}{2}(x-x_0)(x-x_1) \right|$$

$$\le \frac{1}{2} \max_{x \in [x_0, x_1]} |f''(x)| \max_{x \in [x_0, x_1]} |(x-x_0)(x-x_1)|$$

$$h(x) = (x-x_0)(x-x_1)$$
$$= x^2 - (x_0 + x_1)x + x_0 x_1$$

$$h'(x) = 2x - (x_0 + x_1) = 0$$

$$\Rightarrow \quad x^* = \frac{x_0 + x_1}{2}$$

$$|h(x^*)| = \left| \left(\frac{x_0+x_1}{2} - x_0 \right) \left(\frac{x_0+x_1}{2} - x_1 \right) \right|$$

$$= \left| \left(\frac{x_1 - x_0}{2} \right) \left(\frac{x_0 - x_1}{2} \right) \right|$$

$$\frac{x_1 - x_0}{2} = h$$

$$= \frac{h^2}{4}$$

check $h(x)$ at endpoints x_0, x_1

$$|f(x) - P(x)| \leq \frac{1}{2} \max_{x \in [x_0, x_1]} |f''(x)| \cdot \frac{h^2}{4}$$

$$= \frac{1}{8} h^2 \max_{x \in [x_0, x_1]} |f''(x)|$$

[5]

Suppose one seeks a polynomial approximation for e^{-x} for $x \in [0,1]$ using *equispaced* interpolation nodes. Using the result given in Lecture 13 for the error in polynomial interpolation for this particular case, what is the fewest number of points that will ensure

$$\max_{x\in[0,1]} |f(x) - P(x)| \le 1 \times 10^{-6} \ ?$$

$$\max_{x\in[0,1]} |f(x) - P(x)| \le \frac{1}{4}\frac{M}{n+1} h^{n+1}$$

$$M = \max_{x\in[0,1]} |f^{(n+1)}(x)| = 1$$

$$h = 1/n$$

$$f^{(n)}(x) = \pm e^{-x}$$

$$\frac{1}{4}\frac{M}{n+1} h^{n+1} < 1\cdot 10^{-6}$$

Cubic splines

Given: $x_0, \dots, x_n$

$f(x_0), \dots, f(x_n)$

Want (piecewise) $S(x)$ so that:

$S(x)$ is cubic on each subinterval $[x_i, x_{i+1}]$

$S(x)$ is "sufficiently smooth"

$\Rightarrow$ $S(x)$ continuous

$S'(x)$ continuous

$S''(x)$ "

More concretely: Given $x_0 \dots x_n$, $f(x_0) \dots f(x_n)$

we want a_j, b_j, c_j, d_j $(j = 0 \dots, n)$

$$S(x) = \begin{cases} S_0(x) = a_0 + b_0(x - x_0) + c_0(x-x_0)^2 + d_0(x-x_0)^3 & \text{on } [x_0, x_1] \\ S_1(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2 + d_1(x-x_1)^3 & \text{on } [x_1, x_2] \\ \vdots & \\ S_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x-x_{n-1})^2 + d_{n-1}(x-x_{n-1})^3 & \text{on } [x_{n-1}, x_n] \end{cases}$$

$$\boxed{h_j = x_{j+1} - x_j}$$

$$a_j = f(x_j) \qquad j = 0, \dots, n$$

$$a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 = a_{j+1} \qquad j = 0, \dots, n-1$$

$$b_j + 2c_j h_j + 3d_j h_j^2 = b_{j+1}, \quad j = 0, \dots, n-1$$

$$c_j + 3d_j h_j = c_{j+1}, \quad j = 0, \dots, n-1$$

"Natural" cubic spline: $S''(x) = 0$ at a, b

$c_0 = 0$, $c_n = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 \\ & & & & & \\ & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} & \\ & & & 0 & 1 & \end{bmatrix}$$

✓ $(n+1) \times (n+1)$

$$\vec{c} = \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} \qquad \vec{f} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) + \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) + \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \end{bmatrix}$$

Ex 1 Given: $\{(0,1), (1,3), (2,2)\}$
Construct the "natural" cubic spline through these points.

Soln

$a_0 = f(x_0) = 1$

$a_1 = f(x_1) = 3$

$a_2 = f(x_2) = 2$

In this case, $h_j = 1 \ \forall j$

Matrix for c_j:

$$\begin{bmatrix} 1 & 0 & 0 \\ h_0 & 2(h_0+h_1) & h_1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{f} = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) + \frac{3}{h_0}(a_1 - a_0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 0 \\ -9 \\ 0 \end{bmatrix}$$

$c_0 = 0, \ c_1 = -9/4, \ c_2 = 0$

$d_0 = \frac{-3}{4}$

C^2 constraints equations:

$c_0 + 3 d_0 h_0 = c_1 \rightarrow 3 d_0 = -9/4$

$c_1 + 3 d_1 h_1 = c_2 \rightarrow -9/4 + 3 d_1 = 0$

$$\rightarrow d_1 = 3/4$$

Continuity constraint equations:

$$a_0 + b_0 h_0 + c_0 h_0^2 + d_0 h_0^3 = a_1$$

$$a_1 + b_1 h_1 + c_1 h_1^2 + d_1 h_1^3 = a_2$$

$$b_0 = 11/4, \quad b_1 = 1/2$$

$$S_0(x) = 1 + 11/4\,(x-0) + 0\,(x-0)^2 - 3/4\,(x-0)^3$$

$$S_1(x) = 3 + \tfrac{1}{2}(x-1) - \tfrac{9}{4}(x-1)^2 + 3/4\,(x-1)^3$$

$$S(x) = \begin{cases} S_0(x) & x \in [0,1] \\ S_1(x) & x \in [1,2] \end{cases}$$

Continuous:

$$S_0(x_1) = S_1(x_1) \qquad h_0$$

$$S_0(x_1) = a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 + d_0(x_1 - x_0)^3$$