## Today:

- HW5 comments
- Extrapolation
- Numerical integration / quadrature
- Hand back midterms

---

## HW5

1) "diagonally dominant" $A$,

WTS: $A$ is invertible

Suppose $\exists$ $x \in \mathbb{R}^n$ with $x \neq \vec{0}$ s.t.

$$Ax = 0$$

For instance, look at $1^{st}$ component of $\vec{0}$

$$\sum_{j=1}^{n} A_{1j} x_j = 0$$

rearrange terms, absolute values, $\Delta$ inequality

2. Piecewise-linear polynomials
   ↳ "connect the dots"

8. b) $\varepsilon := \left| \frac{\varepsilon_2 - \varepsilon_1}{2} \right|$

→ Goal: find value of $h$ that minimizes error

## Extrapolation

- Big idea: Plug in different "step sizes" to error expression for a given method, find a linear combination to eliminate terms

<u>Ex</u> The backward-difference formula can be expressed as

$$f'(x_0) = \underbrace{\frac{1}{h}\left[f(x_0) - f(x_0-h)\right]}_{\text{method}} + \underbrace{\frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3)}_{\text{error terms}}$$

Use extrapolation to derive an $O(h^3)$ formula for $f'(x_0)$.

<u>Soln</u> 3 constraints:

| | |
|---|---|
| 1 | $f'(x_0)$ |
| 0 | $O(h)$ |
| 0 | $O(h^2)$ |

→ 3 different step sizes

Method: $N(h) = \frac{1}{h}\left[ f(x_0) - f(x_0 - h) \right]$

Plug in stepsizes: $h$, $h/2$, $h/4$

$$f'(x_0) = N(h) + \frac{h}{2} f''(x_0) - \frac{h^2}{6} f'''(x_0) + O(h^3)$$
$$f'(x_0) = N(h/2) + h/4\, f''(x_0) - \frac{h^2}{24} f'''(x_0) + O(h^3)$$

$$f'(x_0) = N(h/4) + h/8\, f''(x_0) - \frac{h^2}{96} f'''(x_0) + O(h^3)$$

$$(a+b+c) f'(x_0) = a N(h) + b N(h/2) + c N(h/4) + \left[ \frac{ah}{2} + \frac{bh}{4} + \frac{ch}{8} \right] f''(x_0) + \left[ -\frac{ah^2}{6} - \frac{bh^2}{24} - \frac{ch^2}{96} \right] f'''(x_0) + O(h^3)$$

(Another option: $h$, $2h$, $4h$)

Enforce constraints:

$$a+b+c = 1$$

$$\frac{ah}{2} + \frac{bh}{4} + \frac{ch}{8} = 0 \rightarrow \frac{a}{2} + \frac{b}{4} + \frac{c}{8} = 0$$

$$-\frac{ah^2}{6} - \frac{bh^2}{24} - \frac{ch^2}{96} = 0 \rightarrow -\frac{a}{6} - \frac{b}{24} - \frac{c}{96} = 0$$

Matrix form:

$\vec{d}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1/2 & 1/4 & 1/8 \\ -1/6 & -1/24 & -1/96 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

A = [1 1 1; 1/2 1/4 1/8; -1/6 -1/24 -1/90];

d = [1; 0; 0]

A\d

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2 \\ 8/3 \end{bmatrix}$$

New method:

$$f'(x_0) = \frac{1}{3} N(h) - 2N\left(\frac{h}{2}\right) + \frac{8}{3} N\left(\frac{h}{4}\right) + O(h^3)$$

# Numerical Integration / Quadrature

Trapezoidal rule:

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a)+f(b)\right) \Big\} \text{ rule}$$

$$-\frac{1}{12} f''(\xi)(b-a)^3 \Big\} \text{ error}$$

Ex Approximate

$$\int_0^{\pi/4} x\sin(x)\,dx$$

using the trapezoidal rule. Find an error bound and compare to the actual absolute error, given that

$$\int_0^{\pi/4} x\sin(x)\,dx \approx 0.1517464$$

Soln: Rule:

$$\int_0^{\pi/4} x\sin(x)\,dx \approx \frac{\pi/4 - 0}{2}\left(0(\sin(0)) + \frac{\pi}{4}\sin(\pi/4)\right)$$

$$= 0.2180895$$

To compute error bound, we want to find

$$\max_{\xi \in [0, \pi/4]} |f''(\xi)|$$

→ compute $f''(x)$ to find max

$$|\sin(x) + \cos(x)| \leq 2$$

**Ex** Find $c_0, c_1, x_1$ so the formula

$$\int_0^1 f(x)\,dx \approx c_0 f(0) + c_1 f(x_1)$$

has the highest possible degree of exactness.

Idea: integration is linear

$\Rightarrow$ only need to check monomials

**Soln**: Plug in monomials:

$f(x)=1$

$$\int_0^1 1\,dx = 1 = c_0 \cdot 1 + c_1 \cdot 1$$

$$1 = c_0 + c_1$$

$f(x)=x$

$$\int_0^1 x\,dx = \frac{1}{2} = c_0(0) + c_1 x_1$$

$$\frac{1}{2} = c_1 x_1$$

$f(x)=x^2$

$$\int_0^1 x^2\,dx = \frac{1}{3} = c_0(0)^2 + c_1 (x_1)^2$$

$$\frac{1}{3} = c_1 (x_1)^2$$

$$x_1 = \frac{2}{3}, \quad c_1 = \frac{3}{4}, \quad c_0 = \frac{1}{4}$$

$$\int_0^1 f(x)\,dx \approx \frac{1}{4} f(0) + \frac{3}{4} f\left(\frac{2}{3}\right)$$

$$\int_0^1 x^3\,dx = \frac{1}{4}, \text{ but}$$

$$\frac{1}{4}(0)^3 + \frac{3}{4}\left(\frac{2}{3}\right)^3 = \frac{2}{9}$$

$\rightarrow$ D.O.E. $= 2$

<u>Ex</u> The quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx c_0 f(-1) + c_1 f(0) + c_2 f(1)$$

is exact for polynomials of degree $\leq 2$.
Find $c_0, c_1, c_2$.