# 1 Functions

**Exercise 1.** Show that $f(x) = \cosh(x)$ is one-to-one on $(0, \infty)$ with range $(1, \infty)$. Note:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

**Solution:** To show that $f(x)$ is one-to-one, we will show that it is strictly increasing on the given domain. We compute that the derivative is

$$f'(x) = \frac{e^x - e^{-x}}{2}$$

Then, if $x$ is in the given domain, this means that $x > 0$. Since $e^x$ is a strictly increasing function, this means that $e^x > 1$. Similarly, since $x > 0$, then $-x < 0$, and since $e^x$ is strictly increasing, $e^{-x} < 1$ and therefore $-e^{-x} > -1$. Then for $x > 0$,

$$\begin{aligned} f'(x) = \frac{e^x - e^{-x}}{2} &> \frac{1 - e^{-x}}{2} \quad \text{since } e^x > 1 \\ &> \frac{1 - 1}{2} \quad \text{since } -e^{-x} > -1 \\ &= 0 \end{aligned}$$

Then since $f'(x) > 0$ for $xin(0, \infty)$, we have that $f(x)$ is strictly increasing, which means that it is one-to-one. To find the range of $f(x)$, notice that

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{e^x + e^{-x}}{2} = \infty$$

and since $f(x)$ is continuous and strictly increasing with $f(0) = 1$, this means that the range of $f(x)$ on $(0, \infty)$ is $(1, \infty)$.

**Exercise 2.** Let $f(x) = \ln(\cos(x) + 2)$.
(a) Where does $f(x)$ have minima? Maxima?
(b) Sketch a graph of $f(x)$.
**Solution:** (a) We need to find $x$ so that $f'(x) = 0$, then check if these are minima or maxima. First, using the chain rule, we get

$$f'(x) = \frac{-\sin(x)}{\cos(x) + 2}$$

Then if $f'(x) = 0$, this means that

$$\frac{-\sin(x)}{\cos(x) + 2} = 0$$

and therefore $-\sin(x) = 0$. Hence $x = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$. To distinguish maxima and minima, we need to check the second derivative of $x$. Using the quotient rule, we get

$$\begin{aligned} f''(x) &= \frac{(\cos(x) + 2)(-\cos(x)) - (-\sin(x))(-\sin(x))}{(\cos(x) + 2)^2} \\ &= \frac{-2\cos(x) - \cos^2(x) - \sin^2(x)}{(\cos(x) + 2)^2} \\ &= \frac{-2\cos(x) - 1}{(\cos(x) + 2)^2} \end{aligned}$$

where we used the fact that $\sin^2(x) + \cos^2(x) = 1$ in the last step. Then, checking the critical points,

$$f''(0) = \frac{-2 - 1}{(1 + 2)^2} < 0 \quad f''(\pi) = \frac{2 - 1}{(1 + 2)^2} > 0$$

and in fact, by the properties of cosine, if $x$ is an even multiple of $\pi$, then $f''(x) < 0$, and if $x$ is an odd multiple of $\pi$, then $f''(x) > 0$. Then we have maxima at $x = 0, 2\pi, -2\pi, \ldots$ and minima at $x = \pi, -\pi, 3\pi, -3\pi, \ldots$.
Plugging in to $f(x)$, we finally get that $f(x)$ has maxima at $(0, \ln(3)), (2\pi, \ln(3)), \ldots$ and minima at $(\pi, 0), (-\pi, 0), \ldots$. This gives us enough information to roughly sketch $f(x)$.

# 2 Limits

**Exercise 3.** Evaluate

$$\lim_{x\to 0}(1+x)^{1/x}$$

**Solution:** As written, the limit is of indeterminate form $1^\infty$, so we need to use some tricks to rearrange the limit into a $0/0$ or $\infty/\infty$ form to apply L'Hopital's rule. Since we have an $x$ in the exponent, this is an indication that we should try to use the properties of the logarithm. By the properties of $e$ and ln,

$$(1+x)^{1/x} = e^{\ln((1+x)^{1/x})}$$

Then if we can find $\lim_{x\to 0}\ln((1+x)^{1/x})$, we can take $e$ to that power to find the original limit.

$$\begin{aligned}\lim_{x\to 0}\ln((1+x)^{1/x}) &= \lim_{x\to 0}\frac{1}{x}\ln(1+x) \quad \text{(property of log)}\\ &= \lim_{x\to 0}\frac{\ln(1+x)}{x}\end{aligned}$$

Since $\lim_{x\to 0}\ln(1+x) = 0$ and $\lim_{x\to 0} x = 0$, we can apply L'Hopital's rule. This gives ue

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} \overset{L-H}{=} \lim_{x\to 0}\frac{\frac{1}{1+x}}{1} = \lim_{x\to 0}\frac{1}{1+x} = 1$$

Then, returning to the original limit, we conclude that

$$\lim_{x\to 0}(1+x)^{1/x} = \lim_{x\to 0}e^{\ln((1+x)^{1/x})} = \lim_{x\to 0}e^1 = e$$

**Exercise 4.** Evaluate

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x$$

**Solution:** As written, the limit is of indeterminate form $1^\infty$, so we need to use some tricks. As in the previous problem, by the properties of $e$ and ln,

$$\left(\frac{x}{x+1}\right)^x = e^{\ln((\frac{x}{x+1})^x)}$$

Then if we can find $\lim_{x\to\infty}\ln((\frac{x}{x+1})^x)$, we can take $e$ to that power to find the original limit. By the properties of log,

$$\lim_{x\to\infty}\ln\left(\left(\frac{x}{x+1}\right)^x\right) = \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right)$$

This limit is of indeterminate form $\infty\cdot 0$, so we need to use another trick. Multiplying by $x$ is the same thing as dividing by $1/x$, so we rewrite this as

$$\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}}$$

Since $\lim_{x\to\infty}\ln\left(\frac{x}{x+1}\right)=0$ and $\lim_{x\to\infty}\frac{1}{x}=0$, we can apply L'Hopital's rule. This means

$$\begin{aligned}
\lim_{x\to\infty}\frac{\ln\left(\frac{x}{x+1}\right)}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{\frac{1}{\frac{x}{x+1}}\cdot\frac{x+1-x}{(x+1)^2}}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty}\frac{\frac{x+1}{x}\cdot\frac{1}{(x+1)^2}}{\frac{-1}{x^2}} \\
&= \lim_{x\to\infty}\frac{-x^2}{x(x+1)} \\
&= \lim_{x\to\infty}\frac{-x^2}{x^2+x} = -1
\end{aligned}$$

To summarize, $\lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = -1$. Returning to our original limit, we conclude that

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} e^{\ln((\frac{x}{x+1})^x)} = \lim_{x\to\infty} e^{-1} = e^{-1}$$

# 3 Inverse Trig Derivatives and Integrals

**Exercise 5.** Find $\frac{d}{dx}\arccos(x)$ using the fact that

$$\cos(\arccos(x)) = x$$

**Solution:** Since

$$\cos(\arccos(x)) = x$$

we take the derivative of both side and use the chain rule to obtain:

$$-\sin(\arccos(x)) \cdot \frac{d}{dx}\arccos(x) = 1$$

Therefore,

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sin(\arccos(x))}$$

There are two main methods to simplifying the right-hand-side.
**Method 1:** Use the Pythagorean trig identity. Since $\sin^2(x) + \cos^2(x) = 1$, we can solve for $\sin(x)$ to get

$$\sin(x) = \sqrt{1 - \cos^2(x)}$$

Using this relationship,

$$\begin{aligned}
\frac{d}{dx}\arccos(x) &= \frac{-1}{\sqrt{1 - \cos^2(\arccos(x))}} \\
&= \frac{-1}{\sqrt{1 - (\cos(\arccos(x)))^2}} \\
&= \frac{-1}{\sqrt{1 - x^2}}
\end{aligned}$$

where in the last step we used the fact that $\cos(\arccos(x)) = x$.
**Method 2:** Use trig substitution. By definition, $\arccos(x)$ is the angle $\theta$ between $-\pi/2$ and $\pi/2$ so that $\cos(\theta) = x$. Then we can consider the right triangle with acute angle $\theta$ so that the side adjacent to $\theta$ has length $x$ and the hypotenuse has length 1, and in this case, $\cos(\theta) = x$. Then by the Pythagorean theorem, the length of the side opposite to $\theta$ has length $\sqrt{1 - x^2}$, so $\sin(\theta) = \sqrt{1 - x^2}$. Since $\theta = \arccos(x)$, this means that $\sin(\arccos(x)) = \sqrt{1 - x^2}$, so

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1 - x^2}}$$

**Exercise 6.** Evaluate
$$\int \frac{dx}{\sqrt{4-9x^2}}$$

**Solution:** The integrand looks a lot like $\frac{1}{\sqrt{1-x^2}}$, which is $\frac{d}{dx}\arcsin(x)$. Let's rewrite our integral to get it closer to this form:

$$\begin{aligned}\int \frac{dx}{\sqrt{4-9x^2}} &= \int \frac{dx}{\sqrt{4(1-\frac{9}{4}x^2)}}\\ &= \int \frac{dx}{2\sqrt{1-\frac{9}{4}x^2}}\\ &= \frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{3}{2}x)^2}}\end{aligned}$$

This is a lot closer to the more familiar form, so let's use substitution. Set $u = \frac{3}{2}x$. Then $du = \frac{3}{2}dx$ and so $dx = \frac{2}{3}du$. Substituting, we get

$$\begin{aligned}\frac{1}{2}\int \frac{dx}{\sqrt{1-(\frac{3}{2}x)^2}} &= \frac{1}{2}\int \frac{\frac{2}{3}du}{\sqrt{1-u^2}}\\ &= \frac{1}{2}\cdot\frac{2}{3}\cdot\arcsin(u) + C\end{aligned}$$

and substituting to write this in terms of $x$, we conclude

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\arcsin\left(\frac{3}{2}x\right) + C$$