1 Arc length and surface area

Exercise 1. Compute the arc length of $y = \ln(\sin(x))$ for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

Exercise 2. Show that if the arc length of $y = f(x)$ over $[0, a]$ is proportional to a, then $y = f(x)$ must be a linear function.

Exercise 3. Find the surface area of the torus obtained by rotating the circle

$$x^2 + (y - b)^2 = r^2$$

around the x-axis.

2 Sequences

Exercise 4. Let $a_n = n\sin(\frac{1}{n})$. Find $\lim_{n\to\infty} a_n$.

Exercise 5. Find the limit of the sequence

$$a_n = \frac{(\ln(n))^3}{n}$$

Exercise 6. Show that the sequence

$$a_n = \frac{1}{\ln(n+2)}$$

converges.

Exercise 7. Suppose $a_n = (\frac{1}{5})^n$, and $L = \lim_{n\to\infty} a_n$. What is the smallest N so that for all $n > N$, $|a_n - L| < 10^-5$?

3 Series

Exercise 8. Find the sum of the following infinite series:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots$$

Exercise 9. Does the series

$$\sum_{n=0}^{\infty} \frac{1}{1+e^n}$$

converge or diverge?

Exercise 10. Does the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{5^{(2n+1)}}{(2n+1)!}$$

converge conditionally, converge absolutely, or diverge?

4 Power series

Exercise 11. Write down the series

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

in summation notation.

Exercise 12. There is a function $f(x)$ so that

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

Find an expression for $f(x)$.

Exercise 13. Show that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, |x| < 1$$

Exercise 14. Find the interval of convergence for the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Use this to show that

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Exercise 15. Find the radius of convergence for the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

and show that the series converges at the right endpoint.

Exercise 16. Show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

5 Solutions

Solution 1. We have $y = f(x) = \ln(\sin(x))$, so

$$f'(x) = \frac{\cos(x)}{\sin(x)}$$

Then

$$\begin{aligned}
AL &= \int_{\pi/4}^{\pi/2} \sqrt{1 + \left(\frac{\cos(x)}{\sin(x)}\right)^2} dx \\
&= \int_{\pi/4}^{\pi/2} \sqrt{\frac{\sin^2(x) + \cos^2(x)}{\sin^2(x)}} dx \\
&= \int_{\pi/4}^{\pi/2} \frac{dx}{\sin(x)} \\
&= \int_{\pi/4}^{\pi/2} \csc(x) dx \\
&= -\ln|\csc(x) + \cot(x)| \big|_{x=\pi/4}^{x=\pi/2} \\
&= -\ln|1| + \ln|\sqrt{2} + 1| \\
&= \ln|\sqrt{2} + 1|
\end{aligned}$$

Solution 2. Let $L(a)$ be the arc length of y over $[0, a]$. Then

$$L(a) = \int_0^a \sqrt{1 + (f'(x))^2} dx$$

By the Fundamental Theorem of Calculus part 2 and taking the derivative with respect to a,

$$L'(a) = \sqrt{1 + (f'(a))^2}$$

Since $L(a)$ is proportional to a, there is some constant real number k so that

$$L(a) = k \cdot a$$

and so $L'(a) = k$. This means that

$$k = \sqrt{1 + (f'(a))^2}$$

for any positive real number a, which means that $f'(x)$ must be a constant. This is only true if f is a linear function.

Solution 3. By symmetry, we can compute the surface area of the outer half of the right half of the torus and multiply by 4 to compute the surface area of the entire object. Then for that portion,

$$y = f(x) = b + \sqrt{r^2 - x^2}$$
$$f'(x) = \frac{1}{2} \cdot \frac{-2x}{\sqrt{r^2 - x^2}} = \frac{-x}{\sqrt{r^2 - x^2}}$$

Then

$$\begin{aligned}
SA &= 4\int_0^r 2\pi(b + \sqrt{r^2 - x^2})\sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\,dx \\
&= 8\pi \int_0^r (b + \sqrt{r^2 - x^2})\sqrt{\frac{r^2 - x^2 + x^2}{r^2 - x^2}}dx \\
&= 8\pi \int_0^r (b + \sqrt{r^2 - x^2})\frac{r}{\sqrt{r^2 - x^2}}dx \\
&= 8\pi \int_0^r \left(\frac{br}{\sqrt{r^2 - x^2}} + r\right) dx \\
&= 8\pi \int_0^r \left(\frac{br}{r\sqrt{1 - (\frac{x}{r})^2}} + r\right) dx \\
&= 8\pi \int_0^r \left(\frac{b}{\sqrt{1 - (\frac{x}{r})^2}} + r\right) dx \\
&= 8\pi rb \arcsin(\frac{x}{r}) + rx\Big|_{x=0}^{x=r} \\
&= 8\pi rb(\frac{\pi}{2} + r^2)
\end{aligned}$$

Solution 4. We have that $a_n = f(n)$ where

$$f(x) = x\sin(\frac{1}{x})$$

Then

$$\lim_{x\to\infty} x\sin(\frac{1}{x}) = \lim_{x\to\infty} \frac{\sin(\frac{1}{x})}{\frac{1}{x}} = 1$$

where we use the fact that

$$\lim_{[\text{something}]\to 0} \frac{\sin[\text{something}]}{[\text{something}]} = 1$$

Then $\lim_{nto\infty} a_n = 1$.

Solution 5. We have that $a_n = f(n)$ where

$$f(x) = \frac{(\ln(x))^3}{x}$$

Then

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln(x))^3}{x} &\overset{LH}{=} \lim_{x\to\infty} \frac{3(\ln(x))^2}{x} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{6\ln(x)}{x} \\
&\overset{LH}{=} \lim_{x\to\infty} \frac{6}{x} = 0
\end{aligned}$$

Then $\lim_{n\to\infty} a_n = 0$.

Solution 6. All of the terms are positive, i.e. $a_n > 0$ for all n. Also, since $\ln(x)$ is an increasing function, a_n is a decreasing sequence. Then since a_n is decreasing and bounded below, a_n converges to a finite limit.

Solution 7. We know that $L = \lim_{n\to\infty} a_n = 0$. We want to find the smallest N so that for all $n > N$,

$$|a_n - L| < 10^{-5}$$

and so plugging in, we want to satisfy

$$\left|\left(\frac{1}{5}\right)^n\right| < 10^{-5}$$

which means

$$5^n > 10^{-5}$$

Taking logs, this means

$$n > \frac{\ln(10^{-5})}{\ln(5)}$$

Solution 8. We can write the series as

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots$$

In summation notation, this is

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$

This looks like a situation where we can use partial fractions. We want to find A and B so that

$$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$$

We find that $A=1, B=-1$, so we can write our series as

$$\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}$$

Let's look at the partial sums. Let

$$S_N=\sum_{n=1}^{N}\frac{1}{n}-\frac{1}{n+1}$$

Then

$$S_N=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{N-1}-\frac{1}{N}\right)+\left(\frac{1}{N}-\frac{1}{N+1}\right)$$

This telescoping series has a lot of cancellations. In particular, we get

$$S_N=1-\frac{1}{N+1}$$

Then

$$\sum_{n=1}^{\infty}\frac{1}{n}-\frac{1}{n+1}=\lim_{N\to\infty}S_N=1$$

Solution 9. This series is *almost* $\sum \frac{1}{e^n}$, which is a geometric series, and we know how to deal with those. Then using the limit comparison test:

$$L = \lim_{n\to\infty} \frac{1/(1+e^n)}{1/e^n} = \lim_{n\to\infty} \frac{e^n}{1+e^n}$$

We can equivalently evaluate $\lim_{x\to\infty} \frac{e^x}{1+e^x}$. Then

$$\lim_{x\to\infty} \frac{e^x}{1+e^x} \overset{LH}{=} \frac{e^x}{e^x} = 1$$

Since L is finite and greater than 0, by the limit comparison test, our series has the same convergence behavior as $\sum \frac{1}{e^n}$. But this is a geometric series with ratio less than 1, so it converges. Therefore the original series converges.

Solution 10. We can first check if it converges absolutely. Since we have both exponents (involving n) and factorials, the ratio test will probably work well here.

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{5^{(2(n+1)+1)}}{(2(n+1)+1)!}}{\frac{5^{(2n+1)}}{(2n+1)!}} &= \lim_{n\to\infty} \frac{5^{(2n+3)}}{5^{(2n+1)}} \frac{(2n+1)!}{(2n+3)!} \\
&= \lim_{n\to\infty} 25 \frac{1}{(2n+3)(2n+2)} \\
&= 0
\end{aligned}$$

So by the ratio test, the series converges absolutely.

Solution 11. Recall that we can write the odd numbers as $2n+1$ (if n starts at 0). Then we can write the series as:

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{(2n+1)}}{(2n+1)!}$$

Solution 12. We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Note that we can write $f(x)$ as

$$f(x) = \sum_{n=0}^{\infty} (-1 \cdot x^2)^n$$

So replacing x with $-1 \cdot x^2$ in the first equation, we get

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-1 \cdot x^2)^n$$

so

$$f(x) = \frac{1}{1+x^2}$$

Solution 13. We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$$

and replacing x with $-x$, we have

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \dots$$

Note that if we integrate both sides, we get

$$\ln(1+x) = C + x - \frac{x^2}{2} + \frac{x^3}{3} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Plugging in $x = 0$, we get that $C = 0$. So

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1$$

Solution 14. Since we have something raised to a power involving n, but only in the numerator, let's use the ratio test. We want $L < 1$, where

$$L = \lim_{n\to\infty} \frac{|x|^{n+1}/(n+1)}{|x|^n/n}$$

Then

$$\begin{aligned} L &= \lim_{n\to\infty} |x| \frac{n}{n+1} \\ &= |x| \end{aligned}$$

Then the series converges if $|x| < 1$, so the radius of convergence is 1. We still need to check the endpoints. When $x = 1$, the series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

which converges by the alternating series test. Alternatively,

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1} - \frac{1}{2n} = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$$

and you can show that this series converges using, for example, the limit test.
When $x = -1$, the series is

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{2n+1} \frac{1}{n}$$

Since $2n + 1$ is always odd, we can rewrite the series as

$$\sum_{n=1}^{\infty} \frac{-1}{n}$$

which diverges. So the interval of convergence is $-1 < 0 \leq 1$. Then we can take $x = 1$ to show the desired form for $\ln(2)$.

Solution 15. Since we have something raised to a power involving n, but only in the numerator, let's use the ratio test. We want $L < 1$, where

$$L = \lim_{n\to\infty} \frac{|x|^{2(n+1)+1}/(2(n+1)+1)}{|x|^{2n+1}/(2n+1)}$$

Then

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{|x|^{2n+3}}{|x|^{2n+1}} \cdot \frac{2n+1}{2n+3} \\ &= \lim_{n\to\infty} |x|^2 \\ &= x^2 \end{aligned}$$

This means we need $|x| < 1$, so the radius of convergence for this series is 1. Now we need to check the right endpoint. For $x = 1$, we consider the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

which converges by the alternating series test, so the series converges at the right endpoint.

Solution 16. The right-hand-side of this is the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

with $x = 1$. Consider the series

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-1\cdot x^2)^n$$

If we integrate this series term by term, we get our original series (plus a constant). But because this is a geometric series, we know that

$$\frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1\cdot x^2)^n$$

If we integrate both sides, we get

$$\arctan(x) = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

Plugging in 0 to calculate C, we get

$$\arctan(0) = C$$

and so $C = 0$. This means

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

We computed in the previous problem that the series converges for $-1 < x \leq 1$. Then we can take $x = 1$ to get

$$\arctan(1) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}$$

In other words,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$