Math 31B Discussion 2C/2D

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Week 0

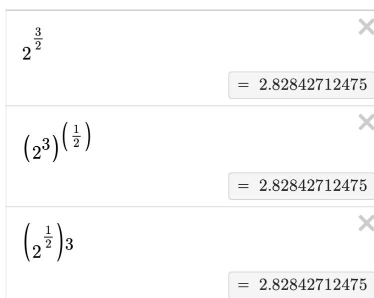
1 Computations with Rational Exponents

Definition 1. Let p, q be integers and $b > 0$. Then we define

$$b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$$

Example 1. Compute $2^{(3/2)}$.

Solution: Using Desmos:



Here, I've included additional rows to represent (using Desmos) some ways you might enter this into a physical calculator.

Example 2. Compute $2^{(4/5)}$.

Solution: Using Desmos:

Example 3. Compute $2^{(23/10)}$.

Solution: There are two main paths here. First, you could just directly compute this. Second, you could use your answers to the previous questions:

$$2^{(3/2)} \cdot 2^{(4/5)} = 2^{(15/10)} \cdot 2^{(8/10)} = 2^{(23/10)}$$

Calculation:

$$2^{\frac{23}{10}} = 4.92457765338$$

$$2.82842712475 \cdot 1.74110112659 = 4.92457765338$$

Fact: For any integers p, q and $b > 0$,

$$b^{p/q} \cdot b^{r/s} = b^{p/q + r/s}$$

Fact: For any $b > 1$, the function

$$f(x) = b^x$$

is strictly increasing. That is, if $x > y$, the $b^x > b^y$.

2 Differentiation Review

Definition 2. The **derivative** of a function f at a point a is the following limit (if it exists):

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

or equivalently:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We can see that these limits are equivalent if we use the substitution $h = x - a$. When the limit exists, we say that f is **differentiable** at a.

Example 4. Suppose $f(x) = x^2$. Find $f'(3)$ using the limit definition of the derivative.

Solution: Using the first definition:

$$\begin{aligned}
\lim_{h \to 0} \frac{(3+h)^2 - (3)^2}{h} &= \lim_{h \to 0} \frac{9 + 6h + h^2 - 9}{h} \\
&= \lim_{h \to 0} 6 + h \\
&= 6
\end{aligned}$$

So $f'(3)$ exists and $f'(3) = 6$. Using the second definition:

$$\begin{aligned}
\lim_{x \to 3} \frac{x^2 - (3)^2}{x - 3} &= \lim_{x \to 3} \frac{x^2 - 9}{x - 3} \\
&= \lim_{x \to 3} \frac{(x+3)(x-3)}{x - 3} \\
&= \lim_{x \to 3} x + 3 \\
&= 6
\end{aligned}$$

Definition 3. A function f is **differentiable** on $D \subseteq \mathbb{R}$ if $f'(x)$ exists for all $x \in D$.

Remark. The "$\subseteq$" symbol means "subset of", and the "$\in$" symbol means "in". $\mathbb{R}$ denotes the real numbers. Also, $\subseteq$ means that the subset *could* be the entire set on the right side of the symbol. So, in words, the above definition is: "A function f is **differentiable** on D, a subset of the real numbers, if $f'(x)$ exists for all x in D.

Example 5. Suppose $f(x) = x^2$. Is f differentiable on $\mathbb{R}$? If so, find $f'(x)$.

Solution: Let's use the first limit definition to check. Suppose x is any real number.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} 2x + h \\
&= 2x
\end{aligned}$$

This is defined everywhere on $\mathbb{R}$, so f is differentiable on $\mathbb{R}$ and $f'(x) = 2x$.