

Math 31B Discussion 2C/2D

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Week 0

1 Computations with Rational Exponents

Definition 1. Let p, q be integers and $b > 0$. Then we define

$$b^{p/q} = \sqrt[q]{b^p} = \left(\sqrt[q]{b}\right)^p$$

Example 1. Compute $2^{(3/2)}$.

Solution: Using Desmos:

The image shows a screenshot of a calculator interface with three rows. Each row contains a mathematical expression on the left and its numerical value on the right. The first row shows $2^{\frac{3}{2}}$ followed by $= 2.82842712475$. The second row shows $(2^3)^{\left(\frac{1}{2}\right)}$ followed by $= 2.82842712475$. The third row shows $\left(2^{\frac{1}{2}}\right)^3$ followed by $= 2.82842712475$. Each row has a small 'x' icon in the top right corner, indicating it's a screenshot of a digital interface.

Here, I've included additional rows to represent (using Desmos) some ways you might enter this into a physical calculator.

Example 2. Compute $2^{(4/5)}$.

Solution: Using Desmos:

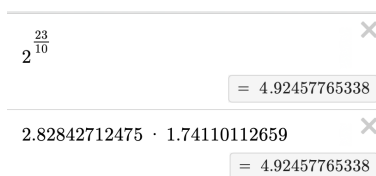
The image shows a screenshot of a calculator interface with one row. It contains the expression $2^{\frac{4}{5}}$ followed by $= 1.74110112659$. There is a small 'x' icon in the top right corner of the input area.

Example 3. Compute $2^{(23/10)}$.

Solution: There are two main paths here. First, you could just directly compute this. Second, you could use your answers to the previous questions:

$$2^{(3/2)} \cdot 2^{(4/5)} = 2^{(15/10)} \cdot 2^{(8/10)} = 2^{(23/10)}$$

Calculation:



A screenshot of a calculator interface. The top line shows the expression $2^{\frac{23}{10}}$ followed by a close button (X) and the result $= 4.92457765338$. The bottom line shows the expression $2.82842712475 \cdot 1.74110112659$ followed by a close button (X) and the result $= 4.92457765338$.

Fact: For any integers p, q and $b > 0$,

$$b^{p/q} \cdot b^{r/s} = b^{p/q+r/s}$$

Fact: For any $b > 1$, the function

$$f(x) = b^x$$

is strictly increasing. That is, if $x > y$, the $b^x > b^y$.

2 Differentiation Review

Definition 2. The **derivative** of a function f at a point a is the following limit (if it exists):

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

or equivalently:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

We can see that these limits are equivalent if we use the substitution $h = x - a$. When the limit exists, we say that f is **differentiable** at a .

Example 4. Suppose $f(x) = x^2$. Find $f'(3)$ using the limit definition of the derivative.

Solution: Using the first definition:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{(3+h)^2 - (3)^2}{h} &= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} \\ &= \lim_{h \rightarrow 0} 6 + h \\ &= 6 \end{aligned}$$

So $f'(3)$ exists and $f'(3) = 6$. Using the second definition:

$$\begin{aligned} \lim_{x \rightarrow 3} \frac{x^2 - (3)^2}{x - 3} &= \lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \\ &= \lim_{x \rightarrow 3} \frac{(x+3)(x-3)}{x-3} \\ &= \lim_{x \rightarrow 3} x + 3 \\ &= 6 \end{aligned}$$

Definition 3. A function f is **differentiable** on $D \subseteq \mathbb{R}$ if $f'(x)$ exists for all $x \in D$.

Remark. The “ \subseteq ” symbol means “subset of”, and the “ \in ” symbol means “in”. \mathbb{R} denotes the real numbers. Also, \subseteq means that the subset *could* be the entire set on the right side of the symbol. So, in words, the above definition is: “A function f is **differentiable** on D , a subset of the real numbers, if $f'(x)$ exists for all x in D .”

Example 5. Suppose $f(x) = x^2$. Is f differentiable on \mathbb{R} ? If so, find $f'(x)$.

Solution: Let's use the first limit definition to check. Suppose x is any real number.

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \rightarrow 0} 2x + h \\ &= 2x\end{aligned}$$

This is defined everywhere on \mathbb{R} , so f is differentiable on \mathbb{R} and $f'(x) = 2x$.