Math 31B Discussion 2C/2D

Joyce Chew

Week 1

1 Differentiation

Definition 1. The **derivative** of a function $f(x)$ is given by the following limit:

$$f'(x) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This definition can be unwieldy, and we recall now some rules for computing derivatives.
Linearity: Suppose that f and g are differentiable. Then:

$$(f+g)' = f' + g' \quad (f-g)' = f' - g'$$

Also, for any constant c, cf is differentiable, and

$$(cf)' = cf'$$

Product rule: Suppose $h(x) = f(x)g(x)$ and f, g are differentiable. Then

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

Quotient rule: Suppose $h(x) = \frac{f(x)}{g(x)}$, f, g are differentiable, and $g(x) \neq 0$. Then

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Chain rule: Suppose $h(x) = f(g(x))$ and f, g and differentiable. Then

$$h'(x) = f'(g(x))g'(x)$$

Product rule: Suppose $f(x) = x^n$. Then

$$f'(x) = nx^{n-1}$$

Derivative of the exponential function: Suppose $f(x) = e^x$. Then

$$f'(x) = e^x$$

Example 1. Suppose $f(x) = x\sin(x)$. Find $f'(x)$.

Solution: First, recall the derivatives of each of the functions in this product:

$$\frac{d}{dx}x = 1, \quad \frac{d}{dx}\sin(x) = \cos(x)$$

Now we apply the product rule, which gives us

$$f'(x) = 1 \cdot \sin(x) + x\cos(x)$$

Example 2. Suppose $f(x) = e^{x\sin(x)}$. Find $f'(x)$.

Solution: Applying the chain rule, we get

$$f'(x) = e^{x\sin(x)}(\sin(x) + \cos(x))$$

Example 3. Suppose $f(x) = \frac{e^x}{x+2}$. Find $f'(x)$.

Solution: Note that $f(x)$ is not defined at $x = -2$, so we'll assume that $x \neq -2$. Now, let's find the derivatives of each of the functions in the quotient:

$$\frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}(x+2) = 1$$

Then using the quotient rule, we get

$$f'(x) = \frac{(x+2)(e^x) - e^x}{(x+2)^2} = \frac{(x+1)e^x}{(x+2)^2}$$

Example 4. Suppose $f(x) = \cos^2(x)$. Find $f'(x)$.

Solution: We apply the chain rule and we get

$$h'(x) = -2\cos(x)\sin(x) = -\sin(2x)$$

Example 5. Suppose $f(x) = \sin^2(e^x - x)$. Find $f'(x)$.

Solution: Applying the chain rule twice, we get

$$f'(x) = 2\sin(e^x - x)\cos(e^x - x)(e^x - 1)$$

2 Logarithms

Definition 2 (Logarithms)**.** Let $b > 0$. Then

$$\log_b(b^x) = x$$

and

$$b^{\log_b x} = x$$

The idea is that logarithms "undo" exponents.

Logarithm properties:

- $\log_b(x) + \log_b(y) = \log_b(xy)$
- $\log_b(x) - \log_b(y) = \log_b(\frac{x}{y})$
- $\log_b(x^n) = n\log_b(x)$
- $\log_b(1) = 0$

Example 6. We can compute the following base-2 logs:

$$\log_2(1) = 0$$

$$\log_2(2) = 1$$
$$\log_2(4) = 2$$
$$\log_2(8) = 3$$
$$\log_2(16) = 4$$
$$\log_2(32) = 5$$

The behavior of $\log_2(x)$ when graphed is typical of all logarithmic functions: sharp increase when $0 < x < 1$, and a rapid "leveling-off" as x goes to ∞. However, the function is always increasing (if $x > y$, then $\log_b(x) > \log_b(y)$.
Change of base: To change between bases for logarithms, we have the following formula:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

One way to remember this is to see the on the left, b is on the bottom and x is on the top, and this is the same on the right.

Example 7. We know that $\log_2(8) = 3$ and $\log_2(4) = 2$. Find $\log_4(8)$.

Solution: Using the change of base formula, we have

$$\log_4(8) = \frac{\log_2(8)}{\log_2(4)} = \frac{3}{2}$$

3 Inverse functions and derivatives

Definition 3. If $f(x)$ is invertible on a given domain, the inverse of $f(x)$ is the function $g(x)$ so that

$$g(f(x)) = x$$

and we write

$$f(x) = f^{-1}(x)$$

Question: When is $f(x)$ invertible?
Answer: $f(x)$ is invertible when $f(x)$ is one-to-one. That is, if $f(x_1) = f(x_2)$, then it **must** be the case that $x_1 = x_2$. Graphically, we can use the horizontal line test. That is, we graph the function and see if a horizontal line (extending off to $+\infty, -\infty$ crosses the graph at multiple points. Sometimes, we will have to restrict the domain of f to ensure that f is invertible on that domain.

Example 8. Let $f(x) = (x-1)^2$. When is $f(x)$ invertible? Find $f^{-1}(x)$.
Solution If we graph $f(x)$, we see that we can choose the domain to be either $x \geq 1$ or $x \leq 1$ for f to be invertible. To find the inverse, we set $y = f(x)$, swap x and y, and solve for y:

$$\begin{aligned}
y &= (x-1)^2 \\
x &= (y-1)^2 \quad \text{(swap)} \\
\sqrt{x} &= y - 1 \quad \text{(solve)} \\
\sqrt{x} + 1 &= y \\
f^{-1}(x) &= \sqrt{x} + 1
\end{aligned}$$

Fact: If $f(x)$ is invertible with inverse $g(x)$ and $f'(a) \neq 0$, then

$$g'(f(a)) = \frac{1}{f'(a)}$$

Equivalently, the book's notation has $f(a) = b$ and

$$g'(b) = \frac{1}{f'(g(b))}$$

Example 9. Suppose $f(x)$ is invertible and $f^{-1}(x) = g(x)$. Suppose we have $f(3) = 5, f'(3) = 10$. What is $g'(5)$?
Solution: In this case (using the above notation), we have $a = 3$ and $b = 5$, so

$$g'(5) = \frac{1}{10}$$

Example 10. Let $f(x) = \sin(x)$.
(a) Find a domain where $f(x)$ is invertible.
(b) Call the inverse $g(x) = \arcsin(x)$. What is $g'(\frac{1}{2})$?
Solution: (a) Graphing $\sin(x)$ and using the horizontal line test, we choose our domain to be $[-\frac{pi}{2}, \frac{\pi}{2}]$. We could also choose translations of this interval, but by convention, this is the domain we use.
(b) Using the notation from before, we have $b = \frac{1}{2}$. To find a, we want to know where $\sin(x) = \frac{1}{2}$ for x in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. On this domain, we have $\sin(\frac{\pi}{6}) = \frac{1}{2}$, so $a = \frac{\pi}{6}$. Next, we need to find $f'(\frac{\pi}{6})$. We have that $f'(x) = \cos(x)$, and so $f'(\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. This is not zero, so know that $g'(\frac{1}{2})$ exists. Finally, using our formula, we get

$$g'(f(\frac{\pi}{6})) = \frac{1}{f'(\frac{\pi}{6})} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$